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THE SIZE OF CHORDAL, INTERVAL AND THRESHOLD SUBGRAPHS[1]

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Given a graph G with n vertices and m edges, how many edges must be in the largest chordal subgraph of G? For $m=n^2/4+1$, the answer is $3n/2-1$. For $m=n^2/3$, it is $2n-3$. For $m=n^2/3+1$, it is at least $7n/3-6$ and at most $8n/3-4$. Similar questions are studied, with less complete results, for threshold graphs, interval graphs, and the stars on edges, triangles, and K_4's.

Introduction

Let G^* be the class of graphs with n vertices and m edges, and H^* a class of graphs. Then there is a number f such that every member of G^* contains a member of H^* with at least f edges, and there is a member of G^* containing no member of H^* with more than f edges. In practice f may be hard to find and we bound it by finding a lower bound f_1 (there must be a member of H^* with at least f_1 edges) and an upper bound f_2 (there need not be a member of H^* with more than f_2 edges).

The questions above are only interesting if the graphs contain sufficiently many edges. The complete bipartite graph $K_{n/2,n/2}$, for example, has no triangles and thus contains no chordal graph larger than a tree ($n-1$ edges), no interval graph larger than an alternating path ($n-1$ edges) and no threshold graph larger than a star ($n/2$ edges). Most of the paper is devoted to the study of graphs with n vertices and at least $n^2/4+1$ edges, that is, graphs that must contain at least one triangle (*dense* graphs). The principal results we obtain in Section 4 may be loosely summarized by (all graphs are assumed dense):

Every graph has a threshold graph of size	$>(1+c)n/2$.
There is a graph with no threshold graph of size	$>(1+.5)n/2$.
Every graph has a star on an edge of size	$=n$.
There is a graph with no star on an edge of size	$>n$.
Every graph has an interval graph of size	$>(1+c)n$.
Every graph has a star on triangle of size	$>(1+c)n$.

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There is a graph with no star on a triangle of size $>(3/2-c)n$.
There is a graph with no interval graph of size $>3n/2-1$.
Every graph has a chordal graph of size $=3n/2-1$ (n even).
There is a graph with no chordal graph of size $>3n/2-1$.

The various constants c, in the main, remain to be determined. Most of these bounds fall into a natural order. However, the numbers for interval graphs and stars on a triangle seem less well determined than the others, and it is by no means clear that the four related entries are in the correct order. (It might be possible to raise each lower bound or lower each upper bound.)

In Section 5 we show that the lower and upper bounds for chordal subgraphs of graphs with $n^2/3$ edges are both $2n-3$ edges, and this is an upper bound for interval graphs. In Section 6 we study graphs with $n^2/3+1$ edges and find a lower bound of $7n/3-6$ and an upper bound of $8n/3-4$ for chordal graphs.

This paper was motivated by two sources, [4] and [5]. Several of the authors were writing a paper [5] concerning coverings and partitions of graphs by chordal and threshold graphs. This required developing tools to find large chordal and threshold graphs in a general graph. One tool we used heavily was the paper [4] in which Erdős and Laskar seek the largest chordal graph in various graphs, in particular in a graph with $n^2/4+1$ edges. We found the tools and examples they suggested extremely valuable; however, we found an error in one of their upper bounds and were able to improve a lower bound. [5] also contains some results related to those here, where the object is to find large graphs of some class (e.g. threshold graphs) within graphs of another class (e.g. chordal graphs).

1. Definitions

All graphs in this paper are finite undirected graphs without parallel edges. The size of a graph is the number of edges. For any definitions omitted here see Golumbic [6].

If $\{A, B, ..., K\}$ is a set of vertices of G, this set of vertices and the set of all edges of G connecting any two of them is the subgraph induced by this set.

A graph is *chordal* (or often *triangulated*; Chapter 4 of [6]) if every cycle of size greater than 3 has a chord (no set of more than 3 vertices induces a cycle). A graph is *interval* (Chapter 8 of [6]) if there is for each vertex A of G an interval (a_A, b_A) of the real line, such that two vertices A and B of G are connected by an edge if and only if the corresponding intervals intersect in the real line. A graph G is *threshold* (Chapter 10 of [6], also [1] and [7]) if there exists a way of labelling each vertex A of G with a nonnegative integer $f(A)$ and there is another nonnegative integer t (the threshold) such that a set of vertices of G induces at least one edge if and only if the sum of their labels exceeds t.

In constructing threshold, interval and chordal graphs we will make use of well-known examples and add certain edges or stars. Thus it will be useful to discuss, as classes of graphs, the star on an edge and the star on a triangle. By the star of a vertex A we mean the set of all vertices joined to A by an edge together with A, and the edges from A to those other vertices. By the star on an edge AB we mean the union of the star of A and the star of B; we call a graph G an *edge-star* or *e-star* if there is in it an edge AB such that G is the star in G on the edge AB. Similarly, the star on a triangle ABC (where AB, BC, and CA are edges of G) is the union of the stars on A, B, and C; a graph G is a *triangle-star* or *t-star* if it can be so constructed. In Section 6 we will similarly use the star on a K_4, the completely connected graph on 4 vertices.

2. Routine Lemmas

We need the following results which are well enough known or obvious enough not to require extensive proof. We prove two just to remind the reader of relevant techniques.

Lemma 2.1. *Every threshold graph is interval* [7]. *Every interval graph is chordal* [6]. ∎

Lemma 2.2. *An induced subgraph of a chordal (interval, threshold) graph is chordal (interval, threshold).* ∎

Lemma 2.3. *If K_t is a clique (completely connected subgraph) in G, then the star on K_t is chordal.* ∎

Lemma 2.4. *The star on any edge is interval.*

Proof. Let the edge be AB. Points in the star may be classified as A_i's (adjacent only to A), B_i's (adjacent only to B), or C_i's (adjacent to both). Let the corresponding intervals be (0, 2) for A, (1, 3) for B, any disjoint subintervals of (0, 1) for the A_i's, of (1, 2) for the C_i's, and of (2, 3) for the B_i's. This can easily be drawn as in Figure 1. ∎

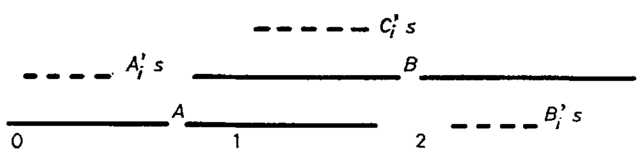

Fig. 1

We need a clear understanding of K_4-free threshold graphs. Consider any one vertex of largest label in such a graph G; it is attached by an edge to every non-isolated vertex of G. Call that vertex A and consider the subgraph H of G induced by all vertices other than A. H is triangle-free (since if it had a triangle, that triangle and A would make a K_4 in G). Pick any vertex B of largest label in H; all edges of H meet B (if there were an edge CD, then $ABCD$ would be a K in G) and all edges of H have both ends connected to A by an edge. Thus every triangle in G has the edge AB and every edge of G is either (a) incident to A, or (b) incident to B and in a triangle with A and B as vertices. We shall call A the root vertex and AB the *root edge* of G. This proves

Lemma 2.5. *Let G be a threshold graph with no K_4 but at least one K_3. Then there is an edge AB of G such that every edge of G is either* (a) *incident to A, or* (b) *of form BC for some C such that ABC is a triangle of G.* ∎

3. Counting Lemmas

It is a well-known principle that, if a graph has average degree a, either there is a vertex of degree considerably higher than a or almost no vertices have degree much below a. It will help us below to make this more formal.

Lemma 3.1. *For every $1>e>0$ there is a $d>0$ (depending only on e) such that if the graph G has n vertices and average degree a, then G either has a vertex of degree exceeding $(1+d)a$ or else G has at most en vertices of degree less than $(1-e)a$.*

Proof. Choose $d<e^2/(1-e)$. Suppose the conclusion is false. Then the en smallest vertices each have degree less than $a(1-e)$ for a total under $ena(1-e)$ and the remaining $n(1-e)$ vertices each have degree less than $a(1+d)$ for a total under $n(1-e)a(1+d)$. Thus the grand total degree for the whole graph is under $an(1+d-e^2-ed)<an$ contradicting the fact that the total degree of G is an. ∎

In the rest of this paper, we will often use $e=.1$, $d=.01$ as an example of values in the above lemma.

Another commonly used calculation is that in the neighborhood of the largest-degree vertex of G there must be a vertex of "not too small" degree. Again, we state this more formally.

Lemma 3.2. *Let G have average degree exceeding $n/2$. Suppose the largest vertex of G has degree $(1+c)n/2$. Then in its neighborhood there must be a vertex of degree at least*

$$\left(1-c+2c^2/(1+c)\right)n/2.$$

Proof. If not, then each of the $(1+c)n/2$ vertices in the neighborhood has degree less than that, while each of the $(1-c)n/2$ points not in that neighborhood has degree not exceeding $(1+c)n/2$; again, the total degree of all vertices works out to no more than $n^2/2$ and we are done. ∎

Finally, it is a common technique to delete a vertex of "low" degree and thereby not lower the average degree of a graph. The following formulations will suffice for us:

Lemma 3.3. *Let G have n vertices and at least $n^2/4+1$ edges. If a vertex of degree less than $n/2$ is deleted, the resulting graph G' has $n-1$ vertices and more than $(n-1)^2/4+1$ edges.* ∎

Lemma 3.4. *Let G have n vertices and at least $n^2/4+1$ edges. If we delete from G two vertices connected by an edge and the total number of edges incident to them is not over $n-1$, then the remaining graph G' has $n'=n-2$ vertices and at least $(n-2)^2/4+1$ edges.* ∎

We will need the following theorems of C. S. Edwards.

Theorem 3.5. ([2]). *If G has n vertices and at least $n^2/4+1$ edges, then there is an edge in G that is on $n/6$ triangles.* ∎

Theorem 3.6. ([3]). *If G has n vertices and m edges, with $m\geqq n^2/3$, then G has a triangle ABC such that the total degree* $\deg(A)+\deg(B)+\deg(C)\geqq 6m/n$. ∎

4. Results

Theorem 4.1. *Let graph G have n vertices and at least* $n^2/4+1$ *edges. Then G contains an e-star with at least n edges.*

Proof. The vertex of largest degree has $(1+c)n/2$ edges, (for a small enough positive c, which may depend on n, since otherwise the total degree of the whole graph can't exceed $n^2/2$). Then by Lemma 3.2 there is in its neighborhood a vertex with $(1-c+2c^2/(1+c))n/2$ edges, that is, at least one more than $(1-c)n/2$. The edge between those two vertices has a star with at least n edges. ∎

An $(m+1)$-regular graph on $n=2m+1$ vertices has more than $n^2/4+1$ edges but every e-star has n edges.

Theorem 4.2. *There is a constant* $c>0$ *such that any graph with n vertices and at least* $n^2/4+1$ *edges contains an interval graph with at least* $(1+c)n$ *edges.*

Proof. We will settle for a proof with $c=.01$. If there is a vertex A with $(1+c)n/2$ neighbors, choose a neighbor B with degree at least $(1-c+2c^2/(1+c))n/2$, which is possible by Lemma 3.2. The e-star on edge AB is an interval graph with at least $(1+c^2/(1+c))n$ edges, and we are done. Otherwise, by Lemma 3.1, we can assume there are fewer than $.1n$ vertices of degree less than $.9n/2$; delete them and find an edge AB in G' lying on at least $n'/6>=.15n$ triangles, which is possible by Theorem 3.5. Pick a triangle in this set and call it ABC. The set consisting of these triangles, the star on A, and the star on C, is an interval graph. To see that this interval graph is large enough, observe that the stars on A and C each contain at least $.45n$ edges and the triangles incorporate at least $.15n$ edges starting at B; $n(.45+.45+.15)=n(1+.05)$ which is more than is needed. This completes the proof. ∎

Theorem 4.3. *There is a constant* $c>0$ *such that any graph with n vertices and at least* $n^2/4$ *edges contains a threshold subgraph with at least* $(1+c)n/2$ *edges.*

The proof is similar to that of Theorem 4.2 and will be omitted. ∎

The following example is motivated directly by one in [4].

Example 4.4. There is a G which contains no threshold graph with as many as $(n/2)(1+1/2+e)$ edges, for any fixed $e>0$ and n sufficiently large. (Hence, the c in the theorem above cannot hope to exceed 1/2.)

Proof. Let A and B be sets of size $(1+c)n/2$ and $(1-c)n/2$ respectively and add all $(1-c^2)n^2/4$ edges from A to B, where $c=1/3$. Now divide A into two equal parts A_1 and A_2 and add $(cn)^2/4+1$ edges between those two parts, distributed as regularly as possible. The degree of a vertex in B is simply $(1+c)n/2$, while each vertex in A_1 has $(1-c)n/2$ edges leading to B and at most just over $((cn)^2/4)/((1+c)n/4)= =nc^2/(1+c)$ edges leading to A_2; so no star can be big enough a threshold graph to disturb us. The graph is tripartite so has no K_4; hence by Lemma 2.5 it has a root vertex and a root edge. We distinguish three cases: (i) root edge from A_1 to A_2, root vertex at either end; (ii) root edge from B to A_1, root vertex in B; (iii) root edge from B to A_1, root vertex in A_1. All other cases are equivalent to these.

(i) The root edge is from A_1 to A_2 and lies on at most $(1-c)n/2$ triangles (the other vertex must be in B); the root vertex has in addition about $nc^2/(1+c)$ edges from A_1 to A_2; total size of this threshold graph is about $(1-c)n+nc^2/(1+c)$ which (since $c=1/3$) is about $(n/2)(1+1/2)$.

(ii) The root edge lies from B to A_1. It lies on about $c^2n/(1+c)$ triangles, and the degree of the vertex at B is at most $(1+c)n/2$ of which $c^2n/(1+c)$ edges have already been counted. The total size of this graph is thus about $(1+c)n/2+ +c^2n/(1+c)$ which is about $(n/2)(1+1/2)$.

(iii) The root edge is from A_1 to B again but we consider also the edges from the vertex in A_1 to B, of which there are $(1-c)n/2$. The total size of the threshold graph is about $2nc^2/(1+c)+(1-c)n/2$ which is about $n/2$. ▮

Theorem 4.2 holds with "interval graph" replaced by "t-star graph". In the first part of the proof, A and B have a common neighbor C and the star on ABC is large enough; in the second part, the star on ABC is large enough. It would be nice to do somewhat more, at least obtaining a larger constant c. However, thus far we have no better lower bound for t-star graphs. Similarly, the only upper bound we have for c for interval graphs is the one we have for chordal graphs.

Example 4.5. There is a G with n vertices and $n^2/4+1$ edges which contains no chordal graph exceeding $(3/2)n-1$ edges. Consider the complete bipartite graph $K_{n/2,n/2}$ plus one edge. Delete a vertex of the one edge; the resulting induced subgraph is still chordal and is contained in a bipartite graph on $n-1$ vertices and thus must be a tree, with at most $(n-1)-1$ edges. Adding back in the star on the deleted point adds at most $(n/2)+1$ edges, for a total of at most $(3/2)n-1$ edges.

Note added in proof: Since this paper was submitted, Genhua Fan has shown (Degree sums for a triangle in a graph, *J. Graph Theory* **12** (1988), 249—263) that in a graph with more than $n^2/4$ edges, there must be a triangle with degree sum $21n/16$, improving the result of [4].

The same upper bound argument applies to interval graphs and to t-star graphs, since both are chordal. We will give an example below which yields a smaller upper bound for t-star graphs.

Example 4.6. In fact, there is a subgraph of the complete bipartite graph plus one edge which attains $(3/2)n-1$ edges and is both interval (hence chordal) and t-star: pick any one triangle containing the one added edge and take the star of that triangle. It is easy to see that the resulting graph has exactly $(3/2)n-1$ edges and is an interval graph.

Example 4.7. There is a G which contains no t-star exceeding $(3/2-e)n$ edges. e can be taken slightly larger than 1/32. This is essentially a restatement of an example of [4].

Proof. Consider the graph of Example 4.4. The graph is tripartite and any triangle has one point in each of B, A_1, and A_2. The total degree of such a triangle is therefore just over $n(3/2-c/2+2c^2/(1+c))$. Letting $c=1/8$ establishes that e can exceed 1/32. Differentiating reveals the best result is obtained when $c=-1+2\sqrt{3/3}$ and a minimum total degree of roughly $n(3/2-.029)$. ▮

Remark. [4] attempts to use this example to obtain an upper bound on the size of chordal graphs. But the graph may well contain a larger chordal graph than it does a t-star. Consider the star on a triangle with one vertex each in B, A_1, and A_2; it includes the star on one edge from A_1 to A_2, which involves only about $2c^2n/(1+c)$ vertices of A_1 and A_2. Thus, there will be many edges between A_1 and A_2 not involving the points of that star; pick a forest from those edges. Adding this forest to the subgraph in the example does not destroy chordality, but may well enlarge it to exceed $3n/2$ edges.

In fact, there is in such cases a larger chordal graph than the largest star on a triangle.

Theorem 4.8. *Let G have $n>3$ vertices and $n^2/4+1$ edges, for n even. Then G contains a chordal graph on at least $3n/2-1$ edges.*

Proof. The theorem is true for $n=4$. Suppose $n=2m>4$; we will prove it by induction on m. Let G be a graph with the smallest even order $2m$ for which the result fails. If G contains a triangle whose star has at least $3m-1=3n/2-1$ edges, we are done since that star is chordal. Suppose not; pick any triangle ABC in G. Some edge of this triangle has a star with less than n edges (suppose the three vertices have degrees a, b, c with a largest; we are guaranteed that $a+b+c-3<3n/2-2$ so $b+c-1<n-1/3$ so edge BC has fewer than n edges in its star). Now using Lemma 3.4 on deleting an edge of low degree, we obtain a smaller graph G' which has $n-2$ vertices and by the induction hypothesis has a chordal graph of size at least $3(n-2)/2-1$. Add to this chordal graph in G' the triangle ABC in G. Since only A at most is in the chordal graph, no new cycles (except ABC itself) are created and the new subgraph is a chordal graph of size at least $3(n-2)/2-1+3=3n/2-1$ in G. ▮

5. Graphs with $n^2/3$ edges

Having discussed at some length the case with $n^2/4+1$ edges, we now present some results on larger size graphs. Throughout this section we suppose that the graph G has n vertices and exactly $n^2/3$ edges. Again, we have the complete answer for chordal graphs. We also offer a few remarks about interval graphs.

Example 5.1. Let G be the complete tripartite graph $K_{n/3,n/3,n/3}$. Then G contains a chordal graph (the star on a triangle) with $2n-3$ edges and contains no chordal graph with more than $2n-3$ edges.

Proof. Clearly G has no K_4 and many triangles. Any triangle has degree sum $3(2n/3)=$ $=2n$ and its star has $2n-3$ edges. To show that there is no larger chordal subgraph, suppose the three vertex sets of G are called H, I, and J and suppose L is a large chordal graph of G. The subgraph of L induced by its vertices in the sets H and I (one of the three "sides" of G) is chordal (since induced) and bipartite (since H and I induce a bipartite graph) and thus a tree with at most $2n/3-1$ edges. Similarly the induced subgraphs of L in the other two "sides" of G have at most $2n/3-1$ edges. Thus L has in total at most $3(2n/3-1)=2n-3$ edges. ▮

The star on a triangle in the graph G above is not an interval graph; it contains as an induced subgraph the non-interval subgraph of figure 7. It is an instructive and non-trivial exercise (which we do not reproduce here) to convince yourself that the star on a triangle in the complete tripartite graph contains no interval graph of size exceeding $5n/3-2$ edges. Nevertheless, G does contain a large interval graph.

Example 5.2. The complete tripartite graph G (above) contains an interval graph with $2n-3$ edges. Number the vertices of G $1, 2, 3, \ldots, n$ in rotation (so part H of G contains vertices $1, 4, 7, \ldots$, part I contains $2, 5, 8, \ldots$, and so on). Connect vertex i to vertices $i-2, i-1, i+1, i+2$ for $3<i<n-2$. It is easy to check that this is an interval graph (vertex i corresponds to the interval $(i, i+2.1)$ for each i) and that it has $2n-3$ edges.

Theorem 5.3. *If G has n vertices and $m \geqq n^2/3$ edges, then it contains a chordal graph (the star on a triangle) containing at least $2n-3$ edges.*

Proof. This is a direct consequence of Edwards' theorem on triangles, Theorem 3.6. It states that G has a triangle with degree sum at least $6m/n \geqq 2n$, hence with a star with at least $2n-3$ edges. ∎

6. Graphs with at least $n^2/3+1$ edges

We learned above that graphs with $n^2/3$ edges must have chordal graphs with $2n-3$ edges and need not have larger ones. We here try to determine how much of a jump there must be when G has more than $n^2/3$ edges, that is, has enough edges that it must contain a K_4.

We conjecture that the correct size of the chordal graph in this case is $8n/3-4$. We have this as an upper bound but our lower bound is at present only $7n/3-6$. We will look at the upper bound first.

Example 6.1. $G=K_{n/3,n/3,n/3}+e$ (the complete tripartite graph on n points, with one edge added) contains a chordal graph (in fact, the star on a K_4) which has $8n/3-4$ edges, and contains no larger chordal graph.

We have not found an easy way to construct an interval graph substantially larger than the one in Example 5.2 for this case.

Theorem 6.2. *If G has n vertices and $m>n^2/3$ edges, then G contains a chordal graph (in fact, a star on a K_4) with more than $7n/3-6$ edges.*

Proof. By Theorem 3.6 again, there is in G a triangle with total degree at least $6m/n>$ $>2n$. There must be a vertex D_1 which with this triangle forms a K_4; for if each of the other $n-3$ vertices was attached to at most 2 of the vertices of the triangle, their total degree could not exceed $2(n-3)+6=2n$. If the dgree of the vertex D_1 is at least $n/3$, then the K_4 has total degree exceeding $7n/3$ and its star has more than $7n/3-6$ edges. So we must consider the case when $\deg(D_1)<n/3$.

Let us delete D_1 from G, yielding a graph G_1 with $n_1=n-1$ vertices and $m_1>n^2/3-n/3$ edges. Again $m_1>n_1^2/3$ and $6m_1/n_1>2n$ (we need, and have, $6m_1/n^1$ greater than $2n$ rather than just greater than $2n_1$). So there is a triangle in G_1 with degree sum exceeding $2n$ and making a K_4 with a vertex D_2. If D_2 has degree at least $n/3$, we are done since the new K_4 has a star with over $7n/3-6$ edges; if not, we delete the D_2 to make a graph G_2.

We can continue indefinitely. After the t-th deletion, the graph G_t has $n_t=n-t$ vertices and $m_t>n^2/3-tn/3$ edges, so (as long as t is less than n) it is easy to check that $m_t>n_t^2/3$ so Edwards' Theorem 3.6 applies and there is a triangle with degree sum at least $6m_t/n_t>2n$, the condition for continuing the induction.

But we obviously cannot continue $n-3$ times: this would leave 3 vertices and at most 3 edges, violating $m_t>n_t^2/3$ which held by induction. This contradiction shows that at some point before that D_t has degree exceeding $n/3$, completing the proof. ∎

References

[1] V. Chvatal and P. Hammer, Aggregation of inequalities in integer programming, *Ann. Discrete Math.*, **1** (1977), 145—162.

[2] C. S. Edwards, A lower bound for the largest number of triangles with a single edge, *manuscript*, (about 1977?).

[3] C. S. Edwards, The largest vertex degree sum for a triangle in a graph, *Bull. London Math. Soc.*, **9** (1977), 203—208.

[4] P. Erdős and R. Laskar, A note on the size of a chordal subgraph, *Proc.* **1985** *Southeastern International Conf. on Graphs, Combinatorics, and Computing*, Utilitas Mathematica Publishing Co, 81—86.

[5] P. Erdős, E. Ordman, and Y. Zalcstein, Coverings and partitions with chordal and threshold graphs, *submitted for publication.*

[6] M. Golumbic, Algorithmic Graph Theory and Perfect Graphs, *Academic Press*, New York, **1980.**

[7] P. Henderson and Y. Zalcstein, A graph theoretic characterization of the PVchunk class of synchronizing primitives, *SIAM J. Comp.*, **6** (1977), 88—108.

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